

# Lattice QCD with purely imaginary sources

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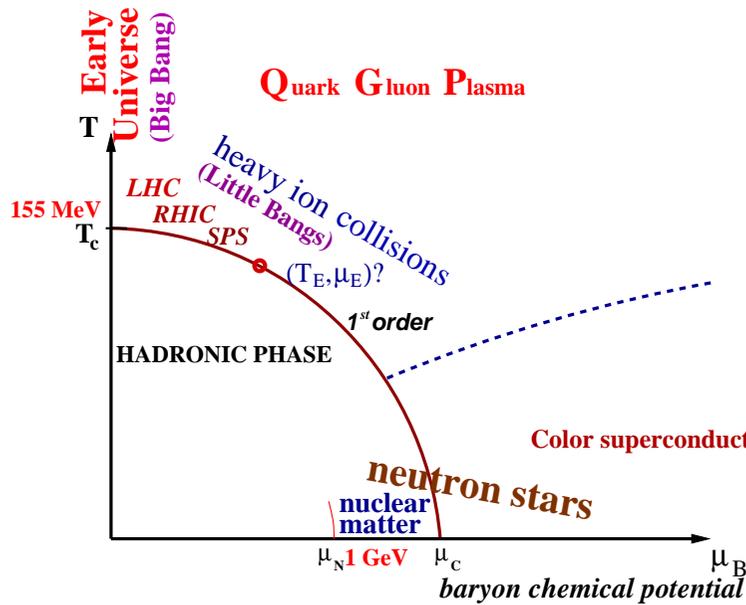
## 1 – Introduction

**Imaginary Sources:** every kind of external source added to the system, such that the path integral measure remains real and positive

- A huge number of examples and applications in Lattice QCD
- Usually aimed at partially avoiding a sign problem present for real sources (e.g., imaginary chemical potentials, imaginary  $\theta$  angle)
- But direct applications as well (e.g., QCD in magnetic background fields)

**I will review such applications and then discuss a selected number of recent results**

# IMAGINARY CHEMICAL POTENTIALS



**QCD at non-zero baryon chemical potential  $\mu_B$  needed to map the QCD phase diagram**

$$Z(T, \mu_B) = \text{Tr} \left( e^{-\frac{H_{\text{QCD}} - \mu_B N_B}{T}} \right)$$

**modified fermion action in the path integral**

**formulation ( $\mu = \mu_B/3$ )**

$$\bar{\psi} (m + \gamma_\nu (\partial_\nu + i(A_\nu - i \delta_{0\nu} \mu))) \psi$$

- Anti-hermiticity properties of  $\gamma_\nu D_\nu$  are lost:  $\det M(\mu)$  **complex** (sign problem)
- **Partial way out** (others: Taylor exp., reweighting, ...): **take  $\mu = i\mu_I$  purely imaginary**
- **On the lattice: temporal links take an additional  $U(1)$  phase:**

$$U_t(n) \rightarrow e^{ia\mu_I} U_t(n) \quad \det M[U, \mu_I] > 0$$

it is a twist by  $\mu_I/T$  in fermionic temporal boundary conditions.

**The partition function is even in  $\mu$ , so it is actually like sending  $\mu^2 \rightarrow -\mu^2$ .**

**Of course,  $\mu = i\mu_I$  is not physical, but quite interesting anyway for various reasons:**

- **Analytic continuation ( $\mu^2 < 0 \rightarrow \mu^2 > 0$ ). Best suited for quantities around  $\mu = 0$ , e.g., for determining the curvature of the critical line (or critical surface)**

M.G. Alford, A. Kapustin, F. Wilczek, 1999; Ph. de Forcrand, O. Philipsen, 2002, 2003, 2007, 2008;

M. D. and M.P. Lombardo, 2003, 2004; P. Giudice and A. Papa, 2004;

V. Azcoiti, G. Di Carlo, A. Galante and V. Laliena, 2005; H.S. Chen, X.Q. Luo, L.K. Wu, 2005, 2007;

M. D., F. Di Renzo, M.P. Lombardo, 2007; F. Karbstein, M. Thies, 2007; S. Conradi and M.D., 2007;

P. Cea, L. Cosmai, M. D. and A. Papa, 2007, 2008, 2009, 2010; M. D. and F. Sanfilippo, 2009;

Y. Shinno and H. Yoneyama, 2009; K. Nagato and A. Nakamura, 2011;

P. Cea, L. Cosmai, M. D., A. Papa and F. Sanfilippo, 2012; P. Cea, L. Cosmai and A. Papa, 2014.

– **applications beyond QCD, e.g. in condensed matter system (J. Braun et al., 2012)**

– **at Lattice 2014: talks by L. Cosmai, M. Mesiti (discussed later)**

- **Reconstruction of the canonical partition function**

$$Z(T, n_q) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\mu_I n_q / T} Z(T, i\mu_I)$$

(A.Hasenfratz, D. Toussaint, 1992; P. de Forcrand, S. Kratochvila, 2006; A. Alexandru et al. 2005, 2010, ...)

– **at Lattice 2014: talk by Y. Taniguchi (canonical approach via winding number expansion)**

- The phase diagram in the  $(T, \mu_I)$  plane may be interesting by its own:
  - Roberge-Weiss transitions and their implications (see later)
  - Phase diagram of dimensionally reduced theories (see, e.g., T. Misumi and T. Kanazawa, arXiv:1405.3113 for QCD with adjoint fermions)
- Results at imaginary  $\mu$  can be used to test and fix the parameters of effective models and numerical approaches, to be used for numerical or analytic predictions at real  $\mu$ . (Y. Sakai, T. Sasaki, H. Kouno, M. Yahiro 2009, 2010, 2012; R. Gatto, M. Ruggieri, 2011; G. Aarts, S. P. Kumar and J. Rafferty, 2010; J. Rafferty, 2011; K. Morita, V. Skokov, B. Friman and K. Redlich, 2012, 2014; K. Kashiwa, T. Sasaki, H. Kouno and M. Yahiro, 2013; R. Pisarski, K. Kashiwa, 2013; M. Ishii et al., 2014)

### Recent developments:

- at Lattice 2014: talk by J. Greensite (simulations at imaginary  $\mu$  to fix the coefficients of effective Polyakov actions, J. Greensite, K. Langfeld, arXiv:1403.5844)
- at Lattice 2014: talk by J. Takahashi (baryon number from imaginary  $\mu$  to fix vector interactions in NJL-model)

## IMAGINARY THETA TERM

**QCD with a topological theta term is another theory with a strong sign problem.**

$$Z(\theta) = \int [dA] e^{-S_{QCD}[A] + i\theta Q[A]} = e^{-V_s f(\theta)/T} \quad Q = \int d^4x \frac{g_0^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x)$$

**Also in this case taking a purely imaginary  $\theta$  is a viable possibility:**

$$\theta = i\theta_I \quad Z(\theta_I) = \int [dA] e^{-S_{QCD} - \theta_I Q}$$

(V. Azcoiti, G. Di Carlo, A. Galante, V. Laliena, hep-lat/0203017; B. Allés and A. Papa 0711.1496; R. Horsley et al. 0808.1428; H. Panagopoulos and E. Vicari, 1109.6815, M.D., F. Negro, 2012, 2013)

## APPLICATIONS

- **Analytic continuation:** also in this case, it is really a  $\theta^2 \rightarrow -\theta^2$  continuation (theory is even in  $\theta$ ).
  - $\theta$ -dependence and  $T - \theta$  phase diagram (V. Azcoiti, G. Di Carlo, A. Galante, V. Laliena, 2002; B. Allés and A. Papa 2007; H. Panagopoulos and E. Vicari, 2011; M.D., F. Negro, 2012, 2013)
  - Electric dipole moment studies (R. Horsley et al. 2008) (discussed later)
  - Effective QED-QCD interactions (M.D., M. Mariti, F. Negro, 2013)
- **Results at imaginary  $\theta$  can be used to test models and numerical methods adopted for real  $\theta$** 
  - **at Lattice 2014:** talk by L. Bongiovanni (comparison of Langevin and direct simulations at imaginary  $\theta$ )

## MOVING FRAMES

Recently, different issues have led to consider QCD formulated in moving (e.g., Lorentz boosted or rotating) frames.

Euclidean path integral measure is positive if movement is in the Euclidean time (no real time evolution) → **one considers imaginary velocities (sign problem otherwise)**

- **Imaginary boost**

$$Z(T, \vec{v}) = \text{Tr} \left( e^{-(H - \vec{v} \cdot \vec{P})/T} \right) \rightarrow Z(T, \vec{\xi}) = \text{Tr} \left( e^{-(H - i\vec{\xi} \cdot \vec{P})/T} \right)$$

corresponding to a spatial shift by  $\vec{\xi}$  in temporal boundary conditions

(L. Giusti, H. Meyer, 2011, 2012, 2013; L. Giusti, M. Pepe, 2014)

– **at Lattice 2014:** talk by M. Pepe (new method to approach thermodynamics)

- **Imaginary angular velocity  $\vec{\Omega} \rightarrow i\vec{\Omega}_I$**

(A. Yamamoto, Y. Hirono, 2013) possible applications to non-central heavy ion collisions and compact stars

## QCD IN EXTERNAL ELECTROMAGNETIC SOURCES

**QCD+QED studies of the e.m. properties of hadrons go back to the early days of LQCD**

- G. Martinelli, G. Parisi, R. Petronzio and F. Rapuano, Phys. Lett. B 116, 434 (1982).

- C. Bernard, T. Draper, K. Olynyk and M. Rushton, Phys. Rev. Lett. 49, 1076 (1982).

**Recent years have seen an increasing activity on the subject, with many studies on vacuum and thermodynamic properties in magnetic backgrounds**

(P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya and M. I. Polikarpov, 2009, 2010; M. Abramczyk, T. Blum, G. Petropoulos and R. Zhou, 2009; M.D., S. Mukherjee, F. Sanfilippo, 2010; M.D., F. Negro, 2011; G. Bali et al., 2011, 2012; E. -M. Ilgenfritz et al. 2012, 2013; G. Bali et al. 2013, 2014; C. Bonati et al. 2013, 2014; C. de Tar, L. Levkova, 2013)

**That is mostly related to the phenomenology of non-central heavy ion collisions: largest magnetic field ever created in a laboratory,  $B \sim 10^{15}$  Tesla, i.e.  $\sqrt{|e|B} \sim 0.5$  GeV (but also interesting for the early Universe and for magnetars)**

An e.m. background field  $a_\mu$  modifies the covariant derivative as follows:

$$D_\mu = \partial_\mu + i g A_\mu^a T^a \quad \rightarrow \quad \partial_\mu + i g A_\mu^a T^a + i q a_\mu$$

in the lattice formulation:

$$D_\mu \psi \rightarrow \frac{1}{2a} \left( U_\mu(n) u_\mu(n) \psi(n + \hat{\mu}) - U_\mu^\dagger(n - \hat{\mu}) u_\mu^*(n - \hat{\mu}) \psi(n - \hat{\mu}) \right)$$

$$U_\mu \in SU(3)$$

$\mathbf{u}_\mu \simeq \exp(i \mathbf{q} \mathbf{a}_\mu(\mathbf{n})) \in U(1)$  similar to an imaginary chemical potential, but the added phase depends both on the link and on the quark charge  $q$ .

in the Euclidean path integral formulation

- $F_{ij} \neq 0 \implies$  non-zero magnetic field
- $F_{0i} \neq 0 \implies$  non-zero imaginary electric field

a real electric field would lead to a sign problem (link phases not in  $U(1)$ )

## 2 – Discussion on some recent results

In the remaining time, I will focus on some selected recent results obtained on the different topics above.

**Apologies** to those whose results will not be discussed further

**Apologies** to those who are doing a great job to provide **real solutions** (not imaginary ones) to various sign problems, which are just mentioned here:

- Lefschetz thimbles and Langevin integration (see talk by D. Sexty)
- Density of states methods
- Formulation in terms of dual variables
- Tensor Renormalization Group techniques
- Worldline approaches
- Effective Polyakov loop models
- Transport properties through dimensional reduction

...

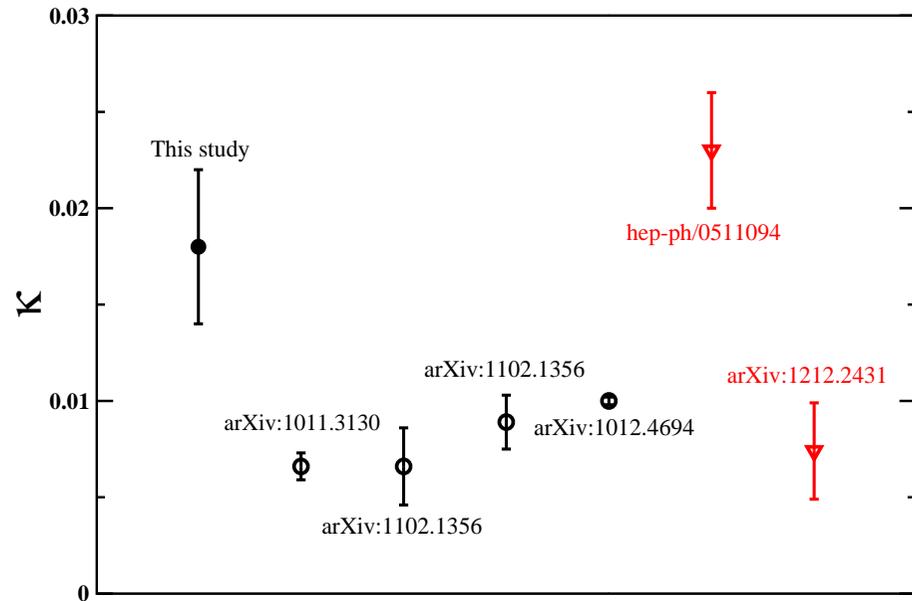
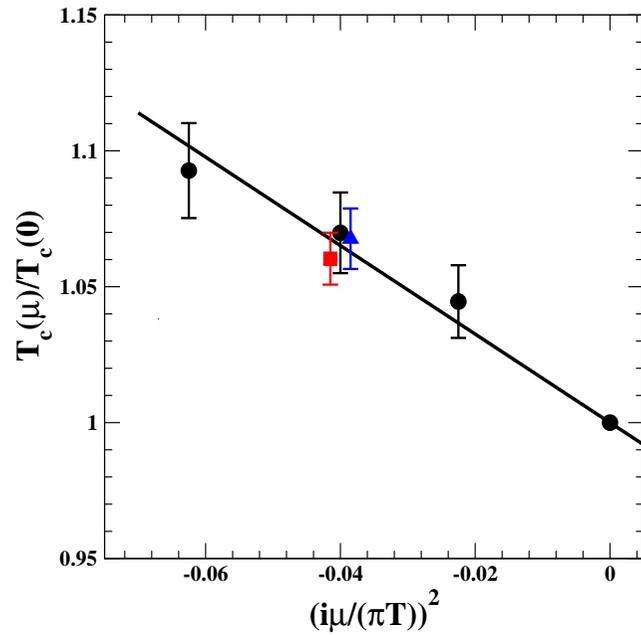
## CURVATURE OF THE CRITICAL LINE FROM ANALYTIC CONTINUATION

The standard way to extract the curvature of the pseudo-critical from analytic continuation is to assume the line is continuous and differentiable around  $\mu^2 = 0$  and consider a lowest order expansion in  $\mu^2$

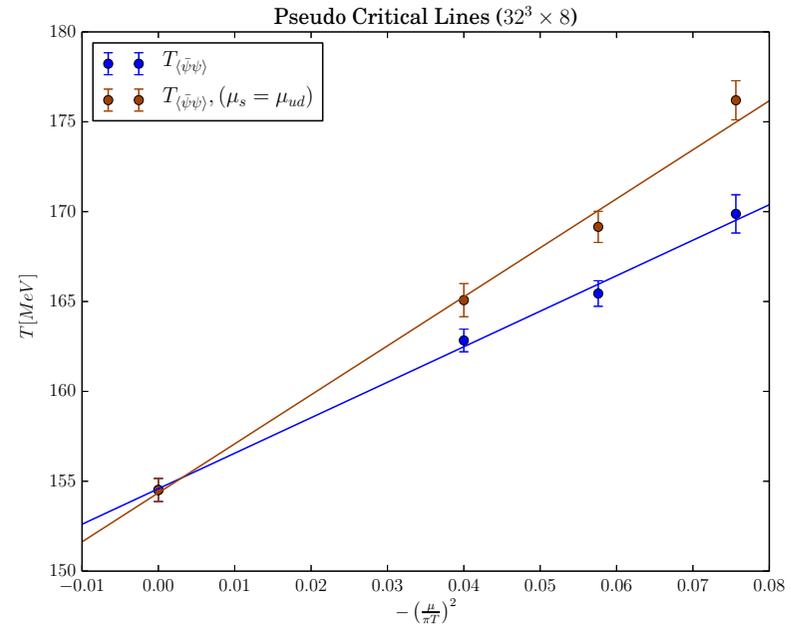
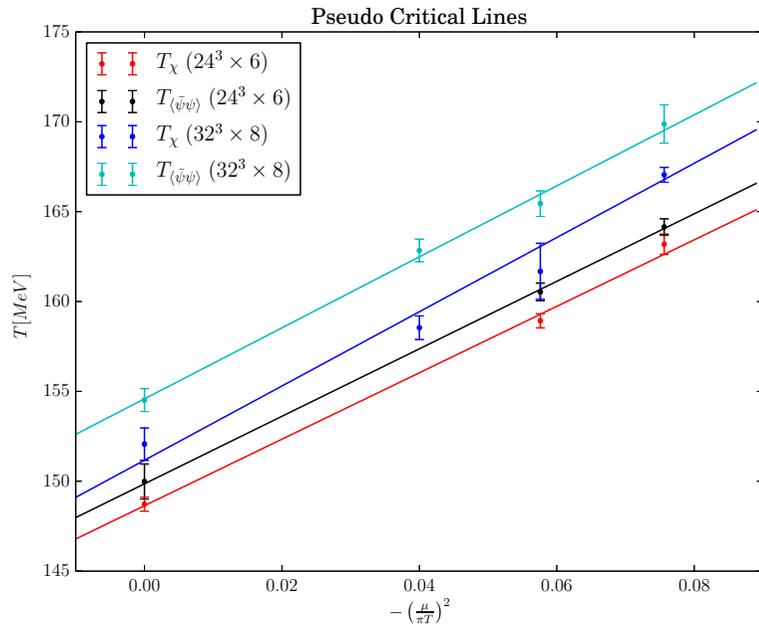
$$\frac{T(\mu_B)}{T_c} \simeq 1 - \kappa \left( \frac{\mu_B}{T(\mu_B)} \right)^2 = 1 - 9\kappa \left( \frac{\mu}{T(\mu)} \right)^2$$

$$\frac{T(\mu_I)}{T_c} \simeq 1 + 9\kappa \left( \frac{\mu_I}{T(\mu_I)} \right)^2$$

- The method and its systematics have been very well studied in models and coarse discretizations of QCD
- Recently, two studies have been done adopting improved discretizations and close to or at the physical point (talks by L. Cosmai and M. Mesiti)
- The issue is fundamental to understand the connection between the pseudo-critical line and the freeze-out curves determined from heavy-ion collisions.



- From P. Cea, L. Cosmai, A. Papa, arXiv:1403.0821 (talk by L. Cosmai)
- HISQ fermions, tree level improved gauge action, line of constant physics at  $m_l/m_s = 1/20$
- $\mu_s = \mu_u = \mu_d$
- **LEFT:** Critical temperature for  $N_t = 6, 8$   $\kappa = 0.018(4)$
- **RIGHT:** Comparison with previous lattice studies (Taylor expansion) and freeze-out curves.



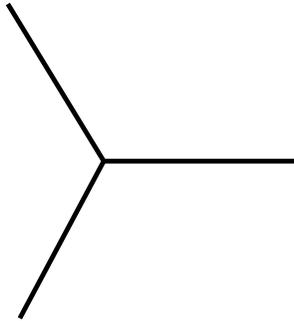
- from C. Bonati, M.D., M. Mariti, F. Negro, M. Mesiti, F. Sanfilippo (talk by M. Mesiti)
- Stout staggered fermions, Symanzik improved action, line of constant physics at  $m_l/m_s = 1/28$
- **LEFT:**  $\mu_u = \mu_d \neq 0$ ;  $\mu_s = 0$ ,  $N_t = 6, 8$   $\kappa \simeq 0.0145$
- **RIGHT:**  $N_t = 8$ , comparison between  $\mu_s = 0$  and  $\mu_s = \mu_{u/d}$ :  $\kappa = 0.0144(7)$  vs  $\kappa = 0.0198(12)$

### COMMENTS

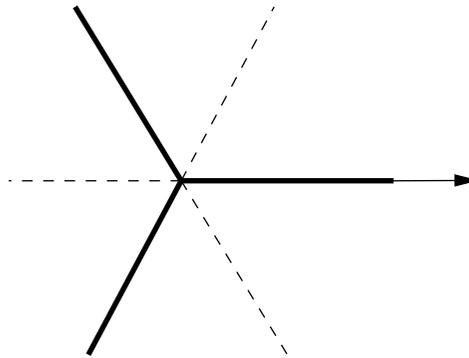
- Inclusion of  $\mu_s$  brings a significant contribution:  $\sim 40\%$
- Curvature from an. continuation generally larger than previous estimates
- continuum extrapolation still needed**

# ROBERGE-WEISS TRANSITION AND ITS IMPLICATIONS

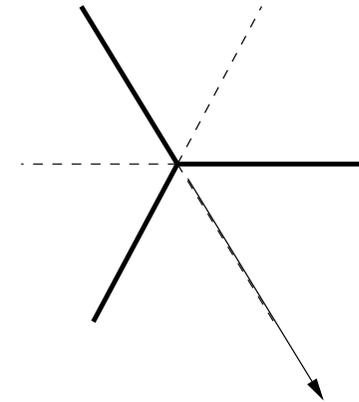
Interesting things happen in the  $T - \mu_I$  plane, in connection with the realization of center symmetry. Imaginary  $\mu = i\mu_I$  rotates fermion temporal b.c. by  $\mu_I/T$



Pure gauge theory  
Exact  $Z_3$  center symmetry  
Spontaneously broken at  $T > T_c$

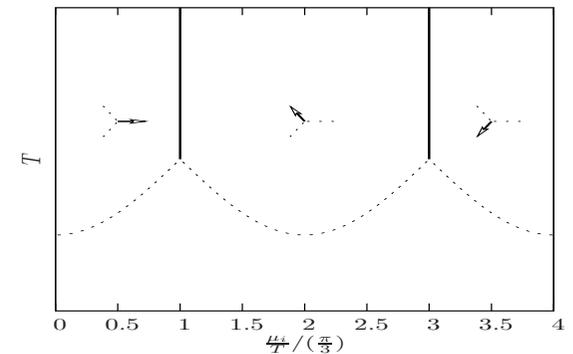


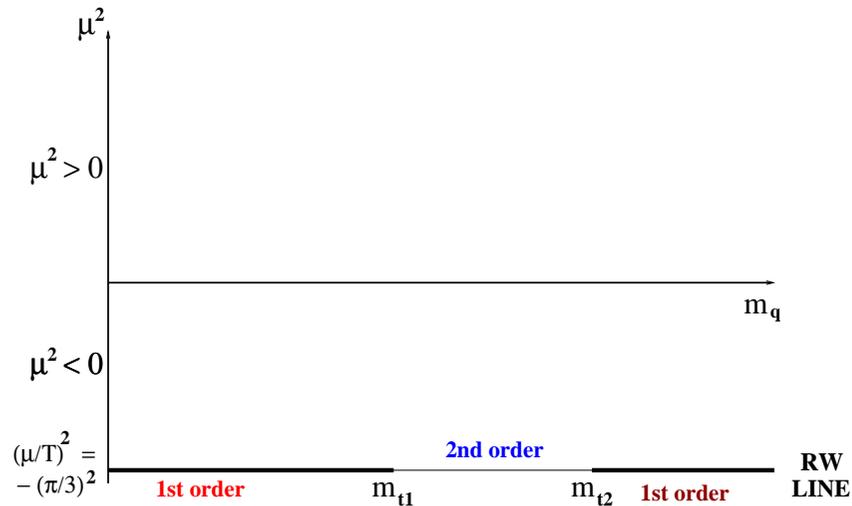
Full QCD,  $\mu = 0$   
Determinant breaks  $Z_3 \rightarrow Z_2$   
No further spontaneous breaking



Full QCD,  $\mu_I/T = (2k + 1)\pi/3$   
Determinant breaks  $Z_3 \rightarrow Z_2$   
 $Z_2$  breaks spontaneously at  $T > T_{RW}$

Even if located in the unphysical  $T - \mu_I$  plane, the Roberge-Weiss transition is the only genuine critical point expected a priori for QCD with generic quark masses!  
Information about the order of the transition is relevant





Early studies have shown the RW transition is first order for light or heavy masses and second order (Ising) in the middle

(M. D., F. Sanfilippo, 2009;  $N_f = 2$  staggered)

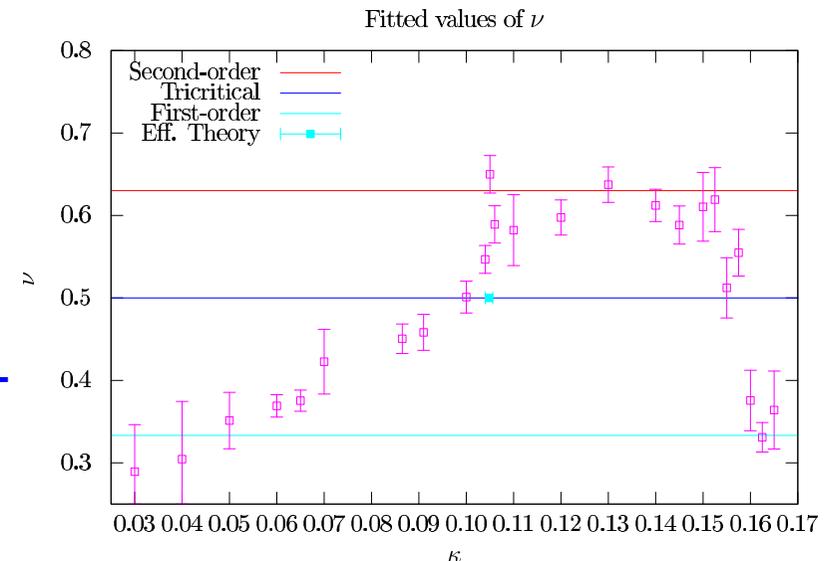
Later studies have clarified and confirmed this interesting scenario

(P. de Forcrand, O. Philipsen, 2010,  $N_f = 3$  staggered)

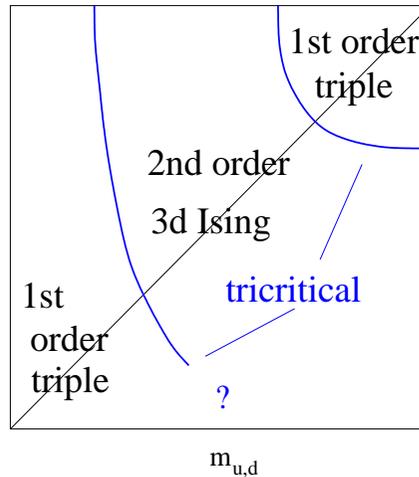
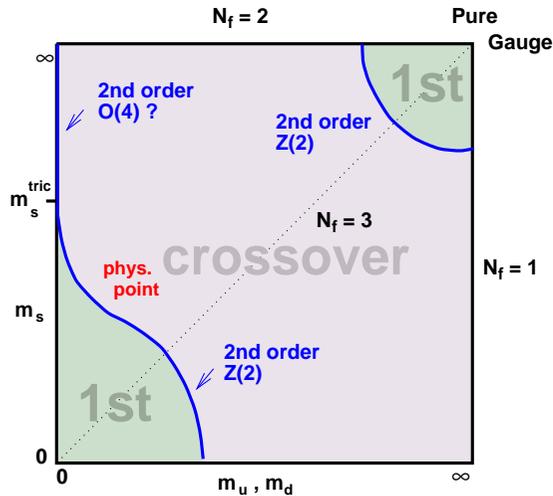
(Bonati, Cossu, M.D., Sanfilippo, 2011,  $N_f = 2$  staggered)

A number of recent studies are trying to confirm this scenario for Wilson fermions

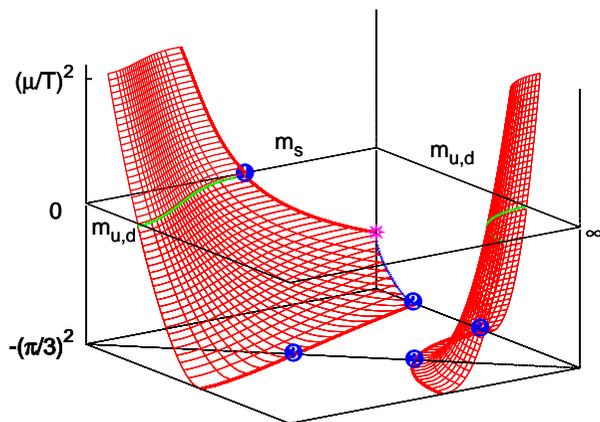
- L.K. Wu and X.F. Meng, 1303.0336, 1405.2425;
- A. Alexandru, A. Li, arXiv:1312.1201
- O. Philipsen, C. Pinke arXiv:1402.0838  
 figure aside from the talk by C. Pinke:  
 critical exponents of the RW transition, confirming the first-second-first order scenario



## Why is the RW transition important?



We have a Columbia plot at a different  $\mu_I$  plane (a plane of real phase transitions) which gives us a different perspective on the original Columbia plot.



tricritical scaling strongly constrains the shape of the critical surface connecting the two planes

(P. de Forcrand, O. Philipsen, 2010).

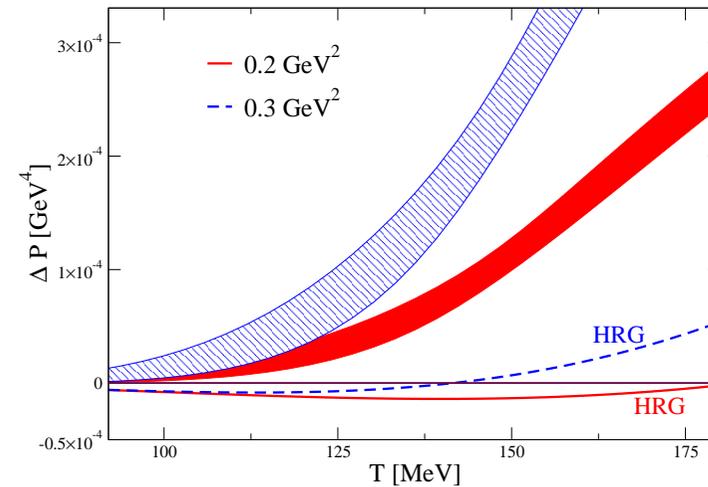
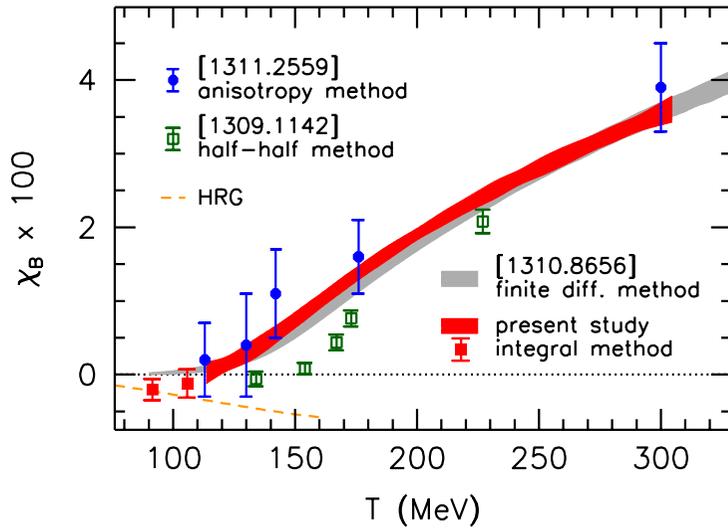
That has given a chance to fix the order of chiral transition for  $N_f = 2$ , at least for poor  $N_t = 4$  unimproved staggered fermions: **first order**.

(Bonati, de Forcrand, M.D., Philipsen, Sanfilippo, 2013)

Finite cut-off effects should be checked, preliminary results on  $N_t = 10$ , stout staggered quarks, close to the RW point, suggest a quite low critical mass: **talk by B. Toth**

## Magnetic properties of strongly interacting matter

- **Diamagnetic or paramagnetic?** Answer is in the free energy for uniform  $B$ :  $f(T, B)$ 
  - \*  $\chi = -\partial^2 f / \partial B^2$      $\chi > 0 \implies$  **PARAMAGNETIC**     $\chi < 0 \implies$  **DIAMAGNETIC**
- **Problem:** on the lattice torus,  $B$  is quantized (like Dirac quantization for monopoles):  
 $qB = 2\pi b(L_x L_y a^2)$  with  $b$  integer. As  $b$  moves away from integer, a visible Dirac string appears, taking the flux away: **cannot derive with respect to a uniform field**
- **Solutions:**
  - **pressure anisotropies** (perturbative coefficients needed) (Bali et al., 2013)
  - **compute**  $\Delta f(B, T) = -(T/V) \log(Z(B, T, V)/Z(0, T, V))$  (Bonati et al., 2013), **best done by thermodynamical integration**  $f(B_2) - f(B_1) = \int_{B_1}^{B_2} \vec{\nabla}_p f \cdot d\vec{p}$ ,  
**integration param. and path are immaterial, just starting and final point count**
    - \* **integrate with respect to (real)  $b$**  (Bonati et al., 2013)
    - \* **integrate forw. and back. in mass at different  $B$  fields** (Bali et al., 2013)
  - **half torus  $+B$ , half torus  $-B$ , no issues with quantization.** Compute standard derivatives at  $B = 0$  (Taylor exp.), check for finite size effects (de Tar, Levkova, 2013)



- Paramagnetic behavior with strongly rising  $\chi$  above  $T_c$  Nice agreement between different groups. Remaining difference likely due to different action.

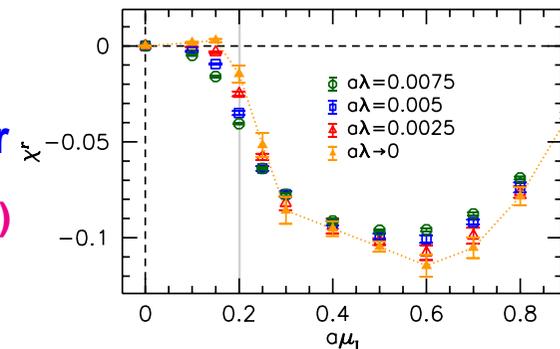
**Left:** from G. Bali et al., 2014, comparing different results

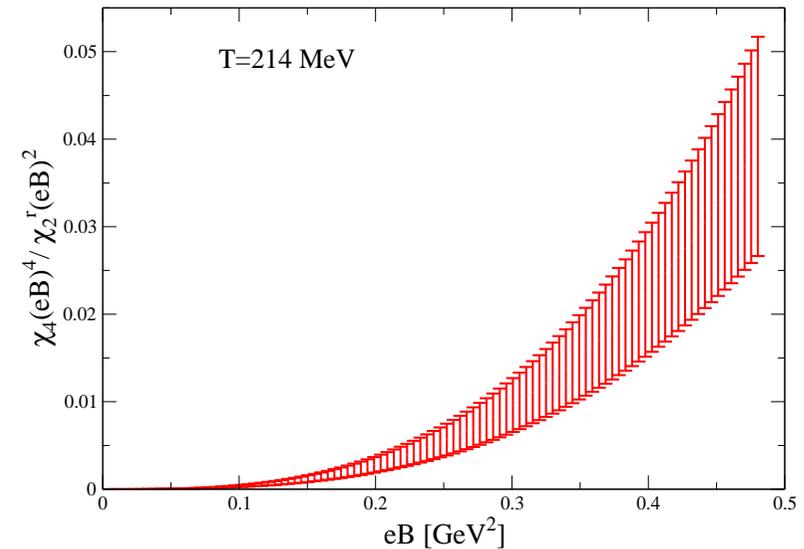
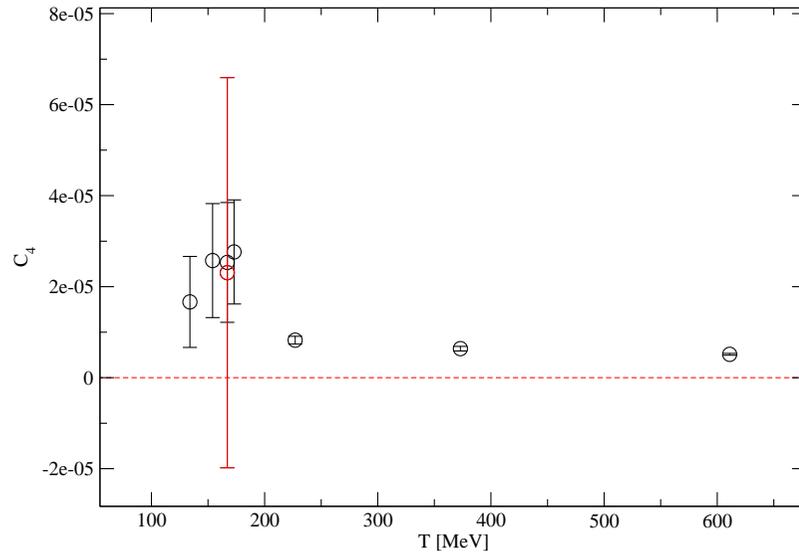
- Lattice results show an unexpected discrepancy with HRG model predictions

**Right:** from C. Bonati et al. 2013, magnetic contribution to the pressure compared to HRG

Predicted diamagnetism at low  $T$  (pion gas) now marginally visible (not yet  $2\sigma$ )

Simulations at finite isospin chemical potential show clear diamagnetism after pion condensation (G. Endrodi, in progress)





- **Non-linear contribution are being computed (talks by Levkova and Mariti)**

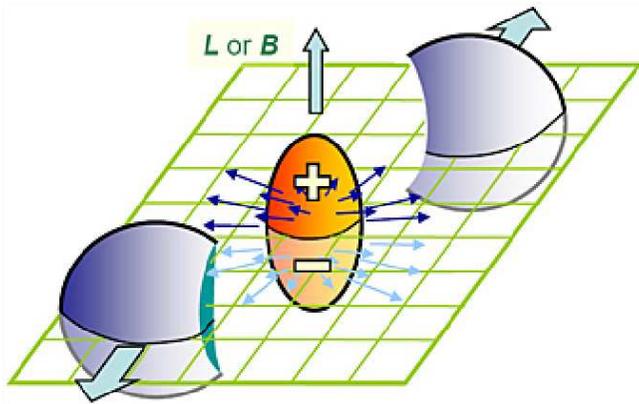
$$\Delta f_r(B, T) = \frac{1}{2} \chi_2^r(T) (eB)^2 + \frac{1}{4!} \chi_4^r(T) (eB)^4 + O((eB)^6)$$

and show that medium response is linear for fields produced in HI collisions ( $eB \sim 0.2 \text{ GeV}^2$ ).

**Left:** unrenormalized quartic derivative of  $f$ , as a function of  $T$ , talk by L. Levkova

**Right:** quartic to quadratic contribution ratio as a function of  $eB$ , talk by M. Mariti

## Chiral Magnetic and related Effects



The separation of electric charges in the presence of magnetic and topological backgrounds has been one of the original motivation for interest in  $B$  effects: (Vilenkin, 1980; Kharzeev, Fukushima, McLerran and Warringa, 2008).

### RECENT RESULTS:

- G. Bali et al., 1401.4141: measurement of the correlation between electric polarization and topological charge in the presence of a magnetic background

$$C_f \equiv \frac{\hat{\tau}_f}{\tau_f} = \frac{\langle q_{top}(x) \cdot \Sigma_{zt}^f(x) \rangle}{\sqrt{\langle q_{top}^2(x) \rangle \langle \Sigma_{xy}^f(x) \rangle}}$$

$C_f \sim O(0.1)$  with little  $T$  dependence. Model predictions:  $C_f \sim O(1)$

- V. Braguta et al, 1401.8095 and talk by V. Braguta:

**Axial Magnetic Effect**  $J_\epsilon = \sigma_{AME} B_5$  (energy flow in the presence of an axial  $B$ )

$\sigma_{AME}/T^2$  steeply rises above  $T_c$ , but magnitude a factor 10 lower than expected.

## Inverse Catalysis and $B$ dependence of $T_c$

### RECENT CONTRIBUTIONS:

- Indications for inverse magnetic catalysis around the transition also from two-color simulations

(E.-M. Ilgenfritz, M. Muller-Preussker, B. Petersson, A. Schreiber, 1310.7876).

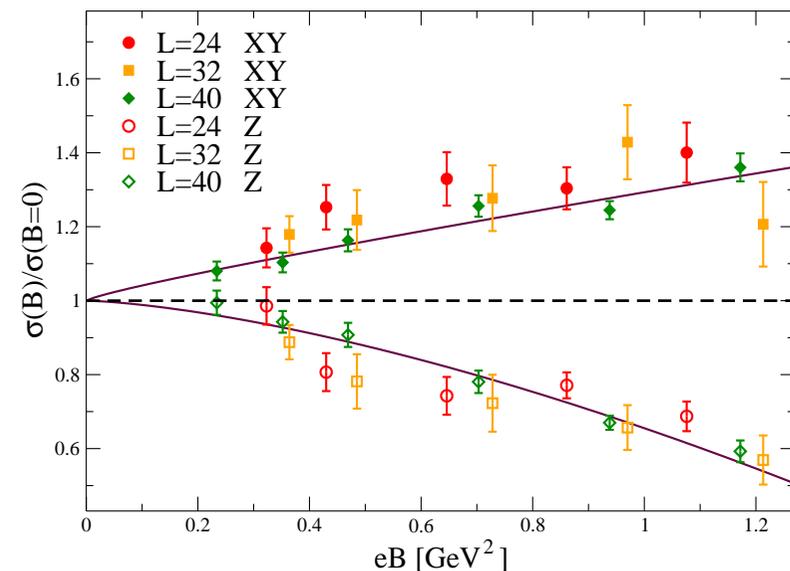
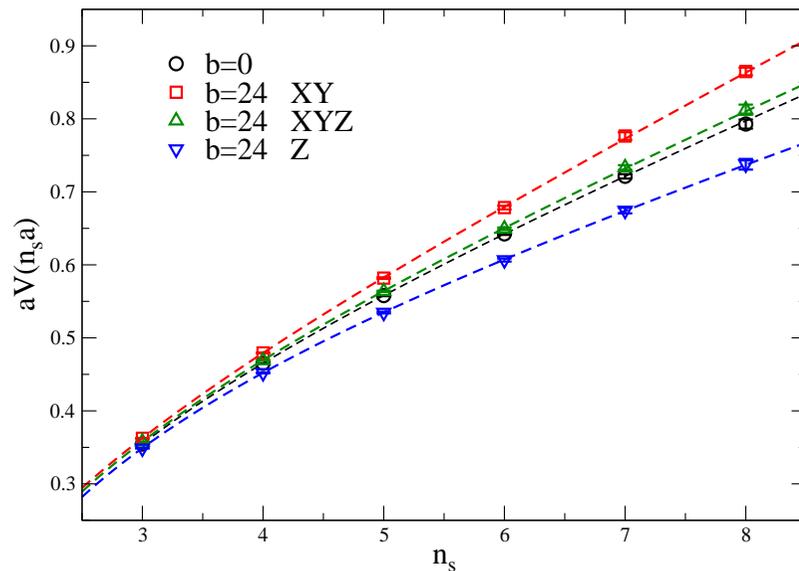
- Evidence for inverse catalysis around  $T_c$  also from simulations with overlap fermions

(Talk by O. Kochetkov).

## Anisotropy of the quark-antiquark potential in a magnetic field

Following various studies demonstrating a significant effect of magnetic fields also on the gluon field distribution, a recent study has shown an influence on the static quark-antiquark potential as well, with the emergence of anisotropies

(C. Bonati et al., 1403.6094, talk by F. Negro)



**LEFT:** the potential gets steeper in the directions orthogonal to  $B$ .

**RIGHT:** the orthogonal (longitudinal) string tension increases (diminishes) as a function of  $B$ .

## $\theta$ dependence and Electric Dipole Moment (EDM)

Upper bounds on  $\theta$  are traditionally based on experimental upper bounds on the EDM of the neutron. That requires a determination of the EDM as a function of  $\theta$

$$d_N(\theta) = d_N^{(1)}\theta + O(\theta^3)$$

**Difficult task.** Usually, a three point function is needed,  $\langle N(\vec{p}_1, s_1) | V_\mu^{EM} | N(\vec{p}_0, s_0) \rangle_\theta$ , with  $\theta$  dependence reconstructed by reweighting configurations sampled at  $\theta = 0$ .

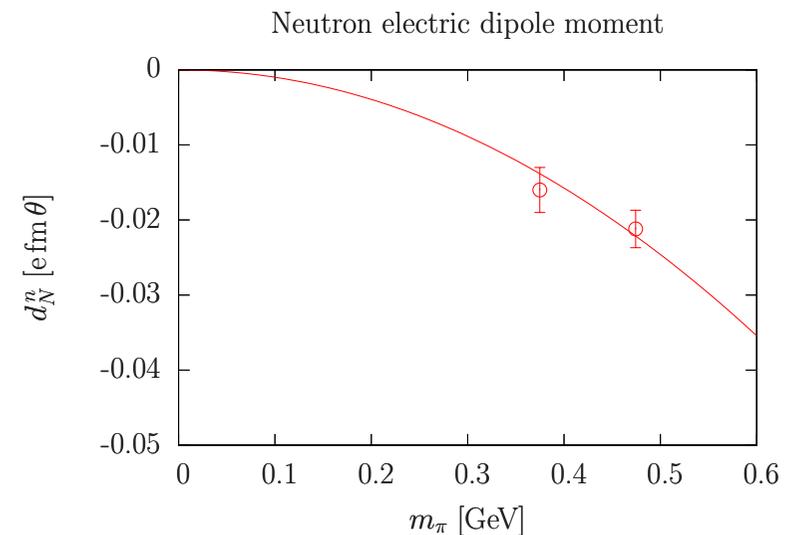
(see, e.g. E. Shintani, T. Blum, A. Soni, T. Izubuchi, PoS-LAT13-298, for recent results for  $N_f = 2 + 1$ )

An interesting alternative is to introduce an imaginary  $\theta$  and perform analytic continuation.

(R. Horsley et al., 0808.1428)

Work in progress for  $N_f = 2 + 1$  (G. Schierholz et al., work in progress and private communication)

leading to an upper bound on  $\theta$  of the order of  $10^{-11}$



## I have a proposal for possible future improvements on the EDM:

- time ago, it was proposed to introduce an external electric background and measure the EDM from the energy difference between states with different spins:

$$\mathcal{E}_+^\theta(E_z) - \mathcal{E}_-^\theta(E_z) = d_N^{(1)}\theta E_z + O(\theta^3)$$

(Shintani et al., PRD75, 0334507 (2007))

- **pros:** needs just a two point function
  - **but:** still needs reweighting in  $\theta$
  - **cons:** real electric field  $\rightarrow$  sign problem
- **proposal:** one could combine imaginary  $\theta = i\theta_I$  and imaginary electric fields  $\vec{E} = i\vec{E}_I$  to perform direct simulations of the two point function

$$\mathcal{E}_+^{\theta_I}(E_{Iz}) - \mathcal{E}_-^{\theta_I}(E_{Iz}) \simeq (i)^2 d_N^{(1)}\theta_I E_{Iz} = -d_N^{(1)}\theta_I E_{Iz}$$

relies on analytic continuation in two different imaginary sources. Idea similar to computation of the CP-odd susceptibility of the QCD vacuum.

(real  $B$  + imaginary  $E \rightarrow$  imaginary  $\theta$ , M.D, M. Mariti, F. Negro, 2013)

### 3 – Conclusions

Lattice simulations of QCD perturbed by external sources can provide a plenty of theoretical and phenomenological information:

- direct determination of relevant physical quantities, like for QCD in external magnetic fields
- indirect information, when imaginary sources are used to avoid a sign problem.

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